Collective generation of quantum states of light by entangled atoms

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We present a theoretical framework to describe the collective emission of light by entangled atomic states. Our theory applies to the low excitation regime, where most of the atoms are initially in the ground state, and relies on a bosonic description of the atomic excitations. In this way, the problem of light emission by an ensemble of atoms can be solved exactly, including dipole-dipole interactions and multiple light scattering. Explicit expressions for the emitted photonic states are obtained in several situations, such as those of atoms in regular lattices and atomic vapors. We determine the directionality of the photonic beam, the purity of the photonic state, and the renormalization of the emission rates. We also show how to observe collective phenomena with ultracold atoms in optical lattices, and how to use these ideas to generate photonic states that are useful in the context of quantum information.

I. INTRODUCTION

In the last years, the field of atomic, molecular and optical physics has witnessed an impressive advance in the development of setups to trap atoms under different conditions, like for example, ions in electromagnetic traps and ultracold atoms in optical lattices. Furthermore, the quantum state of these systems may be engineered by performing quantum operations such as as quantum gates between ions [1], or the excitation of neutral atoms under the dipole blockade [2, 3]. In this way one can create deterministically collective entangled states, like the completely symmetric states with a single excited atom ($|W\rangle$-states) [4]. In addition, those are systems where atoms can be coupled to light in a very controlled way. Since some of the atomic entangled states which may be created in those setups play an important role in the description of the interaction of atomic ensembles with light, an important question arises, namely, can we use our control on the states of trapped atoms to generate useful quantum states of light? If so, which are the properties of such states in terms of photon directionallity, purity, or photon entanglement?

A preliminary study of these ideas was recently presented in [5]. The understanding of this problem requires a theoretical framework to describe the emission of light by collective atomic states under a variety of trapping conditions. Related problems have been indeed a subject of investigation since the seminal work by Dicke [6]. Nevertheless, the experimental systems that we have in mind share a few peculiarities which demand a new approach for their description. First, ultracold atoms are trapped in ordered arrangements such as Coulomb crystals or optical lattices, where the distance between atoms, d_0, is comparable to the wavelength of the light, λ. This situation is very different from Dicke superradiance, where λ is larger than the size of the whole system. It also differs

from the case of crystals, where $\lambda \gg d_0$ as considered, for example, in [7]. Also, we study a situation in which atoms are initialized in a collective state, which is not necesarilly created by the absorption of a photon. Thus, rather than the more traditional description in terms of light scattering, we need a theoretical model of the mapping between atomic collective excitations and photons.

In its more general form the above described situation poses a very complicated many-body problem. A crucial simplification is achieved by considering the low excitation sector of the atomic Hilbert space, which in the Holstein-Primakoff (HP) approximation can be described in terms of bosonic spin-waves [8]. This approach has been used in previous works, for example, to study Dicke superradiance [9], slow propagation of light [10], and atom-light interfaces with atomic vapors [11]. Also, the emission of light by ensembles of harmonic oscillators was studied in [12], although in a different regime of trapping conditions than those considered here.

In this paper we make use of the HP approximation to describe the collective emission of light as a mapping between spin-wave excitations and photons. For atoms placed at fixed positions this mapping is a Gaussian completely positive map [13], and we present a method to get explicit expressions of the photonic modes into which light is emitted. Furthermore, we extend this formalism to study the case of atomic vapors. We obtain the following results: (i) For atoms trapped in a regular lattice, there is a regime in which photons may be emitted in a collimated beam, which requires that $\lambda > 2d_0$. In this regime, there is a renormalization of the emission rates, leading to a classification of the low-excitation atomic Hilbert space in terms of superradiant and subradiant spin-waves. Superradiant states decay with a rate that is enhanced by a factor χ, which depends on the dimensionality of the lattice. In 1D and 3D we determine the values $\chi_{1D} \propto \lambda/d_0$ and $\chi_{3D} \propto (\lambda/d_0)^2(L/d_0)$, respectively, with L the length of the lattice. This effect is related but not equivalent to Dicke superradiance. (ii) In the case of atomic vapors the collective characer in the emission of light is determined by $\chi_{en} \propto N(\lambda/L)^2$, where N is the total number of atoms. The directional regime

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requires $\chi_{\rm en} \gg 1$, and the emission rate is enhanced by $\chi_{\rm en}$. (iii) Some of these effects, like the renormalization of the emission rates and the directionality, could be observed in a relatively simple experiment with atoms in optical lattices. (iv) By making use of coherent effects, photonic entangled states could be generated by trapped ions or neutral atoms. Photons that are generated in this way could be collimated and in a pure state, and thus be useful in the context of quantum information.

II. THEORETICAL FRAMEWORK

Our first task is to describe the collective emission of light by an ensemble of atoms. First, we focus on the situation in which atoms are placed at fixed positions. At the end of the section we discuss the effect of atomic motion.

A. The atom-photon map in the Holstein-Primakoff approximation

We consider an ensemble of N atoms trapped by harmonic potentials with trapping frequency ν, and internal levels forming a Λ-scheme, see Fig. 1. Two ground states $|g\rangle$, $|s\rangle$, are coupled by means of a laser with Rabi frequency $\Omega_{\rm L}$ and wavevector $\mathbf{k}_{\rm L}$, through and auxiliary level $|a\rangle$. We define atomic operators $\sigma_j = |g\rangle_j\langle s|$, $\sigma_j^{\rm ss} = |s\rangle_j\langle s|$, which fulfill the commutation relations,

$$\langle[\sigma_j, \sigma_l^+]\rangle = \delta_{j,l}\left(1 - 2\langle\sigma_j^{\rm ss}\rangle\right). \tag{1}$$

Under the condition that the excitation probability of each atom is low, $\langle\sigma_j^{\rm ss}\rangle \ll 1$, we replace atomic operators by HP bosons, $\sigma_j \to b_j$.

The HP approximation allows us to recast the atom-light Hamiltonian as a bosonic quadratic Hamiltonian. After the adiabatic elimination of the upper level $|a\rangle$, our system is described by

$$H = H_0 + H_{\rm lm} + H_{\rm mot}. \tag{2}$$

$H_0 = \sum_{\mathbf{k}} \omega_k a_{\mathbf{k}}^+ a_{\mathbf{k}}$, with $\omega_k = ck$. We consider, for the shake of clarity, a scalar model for the electromagnetic field, since our conclusions do not change when including the dipole pattern, as we show later. However, the photon polarization can be included straightforwardly in our formalism. $H_{\rm lm}$ is the atom-light interaction Hamiltonian in the rotating wave approximation [14] (we set $\hbar = 1$),

$$H_{\rm lm} = \sum_{j,\mathbf{k}} g_k b_j^+ a_{\mathbf{k}} e^{i(\mathbf{k}-\mathbf{k}_{\rm L})\mathbf{r}_j + i\omega_{\rm L}t} + {\rm h.c.}\,,$$
$$g_k = \frac{\Omega_{\rm L}}{2\Delta}\frac{\hbar\omega_k}{2\epsilon_0 V}d_{\rm ga}. \tag{3}$$

V is the quantization volume of the electromanetic field, ϵ_0 the vacuum permitivity, and d_{ga} the dipole matrix element of the $|g\rangle$-$|a\rangle$ transition. $H_{\rm mot}$ is the Hamiltonian describing the motion of the atoms during the emission process, whose energy scale is given typically by the trapping frequency ν. Eq. (3) is obtained under the assumption $\Delta \gg \Omega_{\rm L}, \nu\bar{n}$, with $\bar{n}$ the mean vibrational occupation number. In the resonant case, $\Delta = 0$, the validity of the adiabatic elimination of $|a\rangle$ requires that the decay rate from $|a\rangle$ to $|g\rangle$, $\Gamma_{\rm ag}$, satisfies $\Gamma_{\rm ag} \gg \Omega_{\rm L}, \nu\bar{n}$, and a similar coupling as (3) is obtained. The spontaneous decay of $|a\rangle$ back to $|s\rangle$ is neglected in the forecoming analysis. This is justified either when collective effects enhance the $|e\rangle$-$|g\rangle$ channel, or by choosing $|e\rangle$-$|g\rangle$ to be a cycling transition, see Appendix D. We deal first with the simplest case, in which atoms are placed at fixed positions and the motion of the atoms can be neglected. This is a good approximation in the Lamb-Dicke regime, $(k_{\rm L}x_0)^2(2\bar{n}+1) \ll 1$, where x_0 is the ground state size in the harmonic trap. The effect of the motion is considered in the last subsection. This work relies on the observation
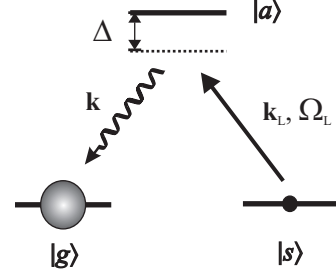

FIG. 1: Lambda scheme.

that $H_0 + H_{\rm lm}$ generates a beam-splitter transformation between atomic states and photons. The evolution of the atom-photon system can thus be understood in terms of a Gaussian completely positive map [13]. In the low excitation regime, any atomic state can be expressed in terms of bosonic excitations,

$$|\Psi\rangle_{\rm at} = \sum_{n_1,\ldots,n_N} \Psi_{n_1,\ldots,n_N} \frac{(b_1^+)^{n_1}}{\sqrt{n_1!}} \cdots \frac{(b_N^+)^{n_N}}{\sqrt{n_N!}}|0\rangle_{\rm at}. \tag{4}$$

We consider the following initial state,

$$|\psi(0)\rangle = |\Psi\rangle_{\rm at}|0\rangle_{\rm ph}, \tag{5}$$

where $|0\rangle_{\rm ph}$ is the photon vacuum. Our goal is to find the photon state $|\Phi\rangle_{\rm ph}$ at a time t longer than the atomic decay time,

$$|\psi(t)\rangle = \mathcal{U}(t)|\psi(0)\rangle = |0\rangle_{\rm at}|\Phi\rangle_{\rm ph}, \tag{6}$$

where $\mathcal{U}(t) = e^{-iHt}$, and H is the total atom-system Hamiltonian. Together with the beam-splitter form of (3) this implies that the problem is reduced to finding the exact form of the transformation,

$$\mathcal{U}(t)b_j^+\mathcal{U}(t)^\dagger = \sum_{\mathbf{k}} g_{j\mathbf{k}}(t)a_{\mathbf{k}}^+ + \sum_l h_{jl}(t)b_l^\dagger. \tag{7}$$

Taking into account that $\mathcal{U}(t)^\dagger|\text{vac}\rangle = 0$, with $|\text{vac}\rangle = |0\rangle_{\text{at}}|0\rangle_{\text{ph}}$, then $g_{j\mathbf{k}}(t)$ can be found by using the relation,

$$g_{j\mathbf{k}}(t) = \langle\text{vac}|a_{\mathbf{k}}\mathcal{U}(t)b_j^+|\text{vac}\rangle = \langle\text{vac}|a_{\mathbf{k}}(t)b_j^+|\text{vac}\rangle, \quad (8)$$

where $a_{\mathbf{k}}(t)$ is the photon operator in the Heisenberg picture. Also,

$$h_{jl}(t) = \langle\text{vac}|b_l\mathcal{U}(t)b_j^+|\text{vac}\rangle = \langle\text{vac}|b_l(t)b_j^+|\text{vac}\rangle, \quad (9)$$

which shows that $h_{jl}(t) = 0$ for t longer than the atomic decay time. $g_{j\mathbf{k}}(t)$ is thus the only interesting term, since it determines the photon mode into which light is radiated. Its determination can be readily done by means of the Heisenberg equation of motion for the photonic operators, which yields

$$g_{j\mathbf{k}}(t) = \quad (10)$$
$$-ig_k e^{-i\omega_k t}\sum_l \int_0^t d\tau e^{-i(\mathbf{k}-\mathbf{k}_{\text{L}})\mathbf{r}_l + i(\omega_k-\omega_{\text{L}})\tau}$$
$$\langle\text{vac}|b_l(\tau)b_j^+|\text{vac}\rangle.$$

This equation is our starting point for the exact determination of the photonic modes.

B. Photonic modes

To find the mapping between atomic and photonic modes, we need to solve the master equation, to calculate the time evolution of the atomic correlator in Eq. (10). In general, this is a difficult many-body problem, but in the bosonic limit it can be described exactly. In terms of HP bosons, the master equation which describes the atomic dynamics reads [15],

$$\frac{d\rho}{dt} = \sum_{i,j}\left(J_{ij}b_j\rho b_i^+ - J_{ij}b_i^+b_j\rho + \text{h.c.}\right), \quad (11)$$

where the J_{ij} include multiple light scattering and dipole-dipole interactions,

$$J_{ij} = \sum_{\mathbf{k}}g_k^2\int_0^\infty e^{i(\omega_k-\omega_{\text{L}})\tau + i(\mathbf{k}-\mathbf{k}_{\text{L}})(\mathbf{r}_i-\mathbf{r}_j)}d\tau. \quad (12)$$

This expression is evaluated by using the identity $\int d\tau e^{i\omega\tau} = \pi\delta(\omega) + i\mathcal{P}(1/\omega)$, which yields the result,

$$J_{ii} = \frac{1}{2}\bar{\Gamma}, \quad (13)$$
$$J_{ij} = \frac{1}{2}\bar{\Gamma}e^{-i\mathbf{k}_{\text{L}}(\mathbf{r}_i-\mathbf{r}_j)}$$
$$\left(\frac{\sin(k_{\text{L}}|\mathbf{r}_i-\mathbf{r}_j|)}{k_{\text{L}}|\mathbf{r}_i-\mathbf{r}_j|} - i\frac{\cos(k_{\text{L}}|\mathbf{r}_i-\mathbf{r}_j|)}{k_{\text{L}}|\mathbf{r}_i-\mathbf{r}_j|}\right)(i\neq j),$$

where

$$\bar{\Gamma} = \frac{1}{3\pi}\left(\frac{\Omega_{\text{L}}}{2\Delta}\right)^2\frac{\omega_{\text{L}}^3}{\epsilon_0 c^3}d_{\text{ga}}^2, \quad (14)$$

is the single atom decay rate. The inclussion of the photon polarization would change the spatial dependence of the couplings J_{ij}. However, as we show later, the collective phenomena would be the same. Note that we are not including the single atom Lamb shift, which may be simply absorved into the laser frequency ω_{L}. Since Eq. (11) is quadratic in bosonic operators, it is readily solved by defining eigenmodes which diagonalize the atomic quantum dynamics in the low-excitation limit,

$$b_j = \sum_n \mathcal{M}_{jn}b_n,$$
$$\left(\mathcal{M}^{-1}J\mathcal{M}\right)_{nm} = J_n\delta_{n,m}. \quad (15)$$

The matrix J_{ij} is not hermitean, and thus canonical commutation relations are not conserved, $[b_n, b_m^+] \neq \delta_{n,m}$. However, the evolution of averages takes a simple form given by

$$\langle b_l(\tau)\rangle = \sum_n \mathcal{M}_{ln}e^{-J_n\tau}\langle b_n(0)\rangle. \quad (16)$$

Thus, the spin-wave dynamics is governed by the eigenvalues J_n. The latter contain the collective decay rates, Γ_n, and the collective energy shifts,

$$\Gamma_n = 2\,\text{Re}\,(J_n), \quad \Delta_n = 2\,\text{Im}\,(J_n). \quad (17)$$

Conservation of the trace under the transformation (15) leads to the following sum rules,

$$\sum_n \Gamma_n = N\bar{\Gamma}, \quad \sum_n \Delta_n = 0. \quad (18)$$

Note that whenever collective effects induce a renormalization of Γ_n, the sum rule implies the existence of super- and subradiant states.

By application of (15, 16), and the quantum regression theorem, we determine the two-time atomic average in (10). In the limit that $t \gg 1/\Gamma_n$ we get

$$g_{j\mathbf{k}}(t) = ie^{-i\omega_k t}g_{\mathbf{k}}\sum_{ln}\frac{\left(\mathcal{M}^{-1}\right)_{nj}e^{-i(\mathbf{k}-\mathbf{k}_{\text{L}})\mathbf{r}_l}\mathcal{M}_{ln}}{i(\omega_k-\omega_{\text{L}}) - J_n}, \quad (19)$$

which yields the explicit form of the atom-photon mapping. By means of this relation it is possible to determine the many-photon state emitted by any initial atomic state like (4).

To get insight of the characteristics of the photonic states emitted by the collective atomic states, we focus from now on, on the mapping to a single photon. To clarify the notation, let us define $|\Psi_n\rangle_{\text{at}}$, the n-spin-wave state with a single excitation,

$$|\Psi_n\rangle_{\text{at}} = \frac{1}{\mathcal{N}_n}\sum_j \mathcal{M}_{jn}b_j^+|0\rangle_{\text{at}},$$
$$\mathcal{N}_n = \sqrt{\sum_j \mathcal{M}_{jn}^*\mathcal{M}_{jn}}. \quad (20)$$

$\mathcal{N}$ has to be included due to the non-hermiticity of J_{ij}. We determine $|\Phi_n\rangle_{\mathrm{ph}}$, the single photon state into which $|\Psi_n\rangle_{\mathrm{at}}$ is mapped,

$$|\Phi_n(t)\rangle_{\mathrm{ph}} = \sum_{\mathbf{k}} \phi_{n,\mathbf{k}}(t) a_{\mathbf{k}}^+ |0\rangle_{\mathrm{ph}},$$

$$\phi_{n,\mathbf{k}}(t) = i\frac{g_k e^{-i\omega_k t}}{\mathcal{N}_n} \sum_l \frac{e^{-i(\mathbf{k}-\mathbf{k}_{\mathrm{L}})\mathbf{r}_l}\mathcal{M}_{ln}}{i(\omega_k - \omega_{\mathrm{L}}) - J_n}. \quad (21)$$

Although these results allow one to solve exactly the problem of collective emission of light including the effects of reabsorption, aditonal insight can be gained by considering the case of a system with periodic boundary conditions. This will be a good approximation for a finite system, provided that the number of atoms in the volume is much larger than in the surface, that is, $L/d_0 \gg 1$. In this case we get the matrices

$$\mathcal{M}_{jn} = \frac{1}{\sqrt{N}} e^{i\mathbf{K}_n \mathbf{r}_j}. \quad (22)$$

The spin-wave state n is then defined like

$$|\Psi_n\rangle_{\mathrm{at}} = \frac{1}{\sqrt{N}} \sum_j e^{i\mathbf{K}_n \mathbf{r}_j} b_j^+ |0\rangle_{\mathrm{at}}. \quad (23)$$

The vector $\mathbf{K}_n$ is the momentum of the collective atomic state. The resulting photonic mode is defined by the following expression,

$$\phi_{n,\mathbf{k}}(t) = ig_k e^{-i\omega_k t} \frac{\frac{1}{\sqrt{N}}\sum_l e^{-i(\mathbf{k}-\mathbf{k}_{\mathrm{L}}-\mathbf{K}_n)\mathbf{r}_l}}{i(\omega_k - \omega_{\mathrm{L}}) - J_n}, \quad (24)$$

such that collective effects and dipole-dipole interactions enter through the dependence of the collective emission rate J_n on the mode number, n. Note that by using periodic boundary conditions, the matrix $\mathcal{M}$ define an unitary transformation, and thus the sets $|\Psi_n\rangle_{n=1,...,N}$, and $|\Phi_n\rangle_{n=1,...,N}$, form an orthogonal basis of spin-waves, and photonic modes, respectively.

C. Angular photon number distribution

We determine now the properties of the emitted photonic modes. In particular, let us define

$$I(\Omega) = \frac{V}{(2\pi)^3} \int_0^\infty \langle a_{\mathbf{k}}^+ a_{\mathbf{k}} \rangle k^2 dk, \quad (25)$$

the average photon number per solid angle. Consider an initial atomic state with a single excitation,

$$|\psi\rangle_{\mathrm{at}} = \sum_j \psi_j b_j^+ |0\rangle_{\mathrm{at}}. \quad (26)$$

The emitted photon distribution is

$$I(\Omega) = \sum_{j,j'} \psi_j^* I_{jj'}(\Omega) \psi_{j'},$$

$$I_{jj'}(\Omega) = \frac{V}{(2\pi)^3} \int_0^\infty g_{j,\mathbf{k}}^* g_{j',\mathbf{k}} k^2 dk. \quad (27)$$

Upon substitution of (19) in the expression for $I_{jj'}(\Omega)$, and under the condition that $\Gamma_n/c \ll 1/L$ (see Appendix A), we get,

$$I_{jj'}(\Omega) = \frac{\bar{\Gamma}}{4\pi} \sum_{n,n'} B(\Omega)_{jn} \frac{1}{J_n^* + \bar{J}_{n'}} B(\Omega)_{j'n'},$$

$$B(\Omega)_{jn} = \left(\mathcal{M}^{-1}\right)_{nj}^* \sum_l \mathcal{M}_{ln}^* e^{i(k_{\mathrm{L}}\mathbf{n}_\Omega - \mathbf{k}_{\mathrm{L}})\mathbf{r}_l}, \quad (28)$$

where $\mathbf{u}_\Omega$ is a unit vector pointing in the direction of the solid angle Ω. In spherical coordinates, Ω is determined by (θ, ϕ), such that $\mathbf{u}_\Omega = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$.

The general recipe for calculating the photon distribution probability involves the following steps: (i) Calculate the coefficients of the master equation and find the eigenvalues and eigenvectors of J_{ij}. (ii) Use Eq. (28) to calculate I_{ij}. (iii) Use the latter to calculate the emission spectrum with the wavefucntion of any given initial atomic state expressed in terms of bosonic spin-waves. In the planewave approximation (22) we can get close expressions for the photon distribution. We focus again in the single photon case, and define $I_n(\Omega)$ as the photon number distribution corresponding to the photonic mode emitted by $|\Psi_n\rangle_{\mathrm{at}}$. By using Eq. (28) we get

$$I_n(\Omega) = \frac{1}{4\pi}\frac{\bar{\Gamma}}{\Gamma_n} \sum_{j,j'} e^{i(\mathbf{k}_\Omega - \mathbf{k}_{\mathrm{L}} - \mathbf{K}_n)(\mathbf{r}_j - \mathbf{r}_{j'})}. \quad (29)$$

This result has a clear interpretation in terms of interference of light emitted by the atomic system [16]. Eq. (29) not only allows one to calculate the angular emission probabilty, but also, due to the normalization condition $\int I_n(\Omega)d\Omega = 1$,

$$\frac{\Gamma_n}{\bar{\Gamma}} = \int d\Omega \frac{1}{4\pi} \sum_{j,j'} e^{i(\mathbf{k}_\Omega - \mathbf{k}_{\mathrm{L}} - \mathbf{K}_n)(\mathbf{r}_j - \mathbf{r}_{j'})}. \quad (30)$$

This expression provides us with a simple way to determine the collective rates under the planewave approximation.

D. Effects of the atomic motion

Finally, we discuss the effect of the atomic motion on the light emission. In the most general case, the inclusion of the motional degrees of freedom poses a very complicated problem which goes beyond the scope of this work. Two time scales determine this problem. First, τ_{mot}, the time scale of the motion of atoms in the trap. In the case of trapped particles, $\tau_{\mathrm{mot}} = 1/\nu$, with ν the trapping frequency. We can extend this discussion to the case of atomic vapors, and consider that in this case, $\tau_{\mathrm{mot}} = L/v$, with L the length of the sample, and v the atom velocity. τ_{mot} is to be compared with the radiative decay time, $1/\bar{\Gamma}$, or more specifically, the set of collective decay times $1/\Gamma_n$. Based on the comparison between these time scales

we define two limits in which the application of our theoretical framework is particularly straightforward.

(i) Slow motion limit, $\tau_{\text{mot}} \gg 1/\Gamma_n$. Since the emission process is much faster than the motion of the particles, we can assume that atomic positions are frozen. The system is in the initial state

$$|\psi(0)\rangle = |\Psi\rangle_{\text{at}}|0\rangle_{\text{ph}}|\Psi_{\mathbf{m}}\rangle_{\text{mot}}, \qquad (31)$$

where $\Psi_{\text{m}}(\mathbf{r}_1, \ldots, \mathbf{r}_N)$ is the initial wavefunction in terms of the atomic positions, which in this limit does not evolve during the emission time. The atom-photon mapping can be still applied to this system by solving it for each value of the atomic positions,

$$|\Psi\rangle_{\text{at}}|0\rangle_{\text{ph}} \rightarrow |0\rangle_{\text{at}}|\Phi^{\mathbf{r}_1, \ldots, \mathbf{r}_N}\rangle_{\text{ph}}, \qquad (32)$$

where $|\Phi^{\mathbf{r}_1, \ldots, \mathbf{r}_N}\rangle_{\text{ph}}$ is the photonic state obtained under the assumptions that ions are located at positions $\mathbf{r}_1, \ldots \mathbf{r}_N$. Any photonic observable, O, is then obtained upon averaging with respect to the wavefunction Ψ_{m}, for example,

$$\langle O \rangle = \sum_{\mathbf{r}_1, \ldots, \mathbf{r}_N} \langle \Phi^{\mathbf{r}_1, \ldots, \mathbf{r}_N} | O | \Phi^{\mathbf{r}_1, \ldots, \mathbf{r}_N} \rangle |\Psi(\mathbf{r}_1, \ldots, \mathbf{r}_N)|^2. \qquad (33)$$

This method can be readily extended to the case of a mixed motional state.

(ii) Fast motion limit, $\tau_{\text{mot}} \ll 1/\Gamma_n$. In this case, we are assuming that trapped atoms move along the sample in a time that is smaller than the emission time. This case is particularly relevant, since it describes hot atomic vapors. We notice first that condition $\tau_{\text{mot}} \ll 1/\Gamma_n$, implies that the atomic positions are not correlated with the atomic operators in Eq. (10). Thus, we can describe the atomic radiative decay indepently of the evolution of atomic positions $\mathbf{r}_l$ in (10). This can be done by using a master equation with averaged coefficients,

$$J_{ij} = \pi \sum_{\mathbf{k}} g_k^2 \, \delta(\omega_k - \omega_{\text{L}}) \langle e^{i(\mathbf{k}-\mathbf{k}_{\text{L}})(\mathbf{r}_i - \mathbf{r}_j)} \rangle_{\text{mot}}, \qquad (34)$$

In the case of an harmonic trap, $\mathbf{r}_i$ are position operators, and $\langle \ldots \rangle_{\text{mot}}$ is the average with the atomic motional state. This situation can be extended to describe hot atomic vapors, by replacing the atomic positions, $\mathbf{r}_i$, by a set of random variables with a given probability distribution $\rho_{\text{m}}(\mathbf{r}_1, \ldots, \mathbf{r}_N)$, and performing the corresponding average to get J_{ij}.

Once the atomic dynamics is solved, our results on the atom-photon mapping can be used to describe the emitted photons. A general atomic state is mapped now into a mixed photonic state. For example, consider the atomic state $|\Psi_m\rangle_{\text{at}}$, defined by Eq. (20), which is obtained by the diagonalization of the master equation with (61). The atom-photon mapping yields the following photon density matrix after the emission process,

$$\rho^{\text{ph}}(t) = \sum_{\mathbf{k}, \mathbf{k}'} \langle \phi_{n,\mathbf{k}}(t)\phi_{n,\mathbf{k}'}^*(t) \rangle_{\text{mot}} a_{\mathbf{k}}^+ |0\rangle_{\text{ph}} \langle 0|a_{\mathbf{k}}, \qquad (35)$$

and this expression is easily generalized to the multiphoton case. Note that the solution of the fast motion limit seems similar to the case of slow motion. However, the crucial difference is that in the fast case, we are allowed to solve the radiative emission problem, and to perform subsequently the spatial average in (35). We will study in more detail this situation later in the case of an the collective emission properties of atomic ensembles.

III. ATOM-PHOTON MAPPING IN A SQUARE LATTICE

The situation in which atoms are arranged in a crystal is found in experimental setups such as ultracold atoms in optical lattices and Coulomb crystals of trapped ions. The results presented in the previous section are applied here to study the collective emission process in these systems. We obtain analytical results by using the plane-wave approximation.

A. General Discussion

Let us study for concreteness the case of one (1D) or three dimensional (3D) square lattices, although our results are easily generalized to different lattice geometries. Assuming periodic boundary conditions, the allowed wavevectors are

$$\mathbf{K_n} = \frac{2\pi}{d_0} \sum_{\alpha} \frac{n_\alpha}{N_\alpha} \hat{\alpha}. \qquad (36)$$

In the 1D case we consider that the atom chain is aligned in the z direction. Thus, in (36) and the forecoming expressions, α runs over $\alpha = z$ in 1D, and $\alpha = x, y, z$ in 3D. $\hat{\alpha}$ is a unit vector in the direction α, and N_α is the number of atoms along $\hat{\alpha}$. Considering, for concreteness, the case of even N_α, each wavevector is determined by the set of integers $n_\alpha = -N_\alpha/2, \ldots, N_\alpha/2 - 1$. By applying Eq. (29) we determine the photon number angular distribution for the photonic state emitted by a spin-wave excitation with momentum $\mathbf{K_n}$,

$$I_{\mathbf{n}}(\Omega) = \frac{1}{4\pi} \frac{\bar{\Gamma}}{\Gamma_{\mathbf{n}}} \qquad (37)$$
$$\frac{1}{N} \prod_{\alpha} \frac{\sin^2((k_{\text{L}}\mathbf{u}_\Omega^\alpha - \mathbf{k}_{\text{L}}^\alpha - \mathbf{K_n}^\alpha)d_0 N_\alpha/2)}{\sin^2((k_{\text{L}}\mathbf{u}_\Omega^\alpha - \mathbf{k}_{\text{L}}^\alpha - \mathbf{K_n}^\alpha)d_0/2)}.$$

$I_{\mathbf{n}}(\Omega)$ shows a series of diffraction maxima at solid angles Ω at which the $\sin^2$-function in the denominator vanishes. Note that by including the photon polarization, we would have got an additional function of Ω multiplying the photon distribution $I_n(\Omega)$, which would correspond to the single atom dipole pattern. The latter would induce the suppression of diffraction peaks, if they are in a direction forbidden by the dipole pattern. Since we are specifically interested on collective effects we do not consider this effect in the discussion that follows.

To get a quantitative description of the photon distribution we notice first that

$$\frac{1}{N}\frac{\sin^2(Nx)}{\sin^2(x)} \approx \sum_m f_N(x - m\pi) \quad \text{if} \ (N \gg 1), \quad (38)$$

where the function $f_N(x)$ describes the shape of each of the diffraction peaks,

$$\begin{aligned} f_N(x) &= N\text{sinc}^2(Nx), \quad -\pi/2 < x < \pi/2, \\ f_N(x) &= 0, \quad \text{otherwise.} \end{aligned} \quad (39)$$

Thus in the limit $N_\alpha \gg 1$, one can approximate the emission probability as a sum over Bragg scattering contributions,

$$\begin{aligned} I_\mathbf{n}(\Omega) &= \sum_\mathbf{m} I_\mathbf{n}^{[\mathbf{m}]}(\Omega), \\ I_\mathbf{n}^{[\mathbf{m}]}(\Omega) &= \frac{1}{4\pi}\frac{\bar{\Gamma}}{\Gamma_\mathbf{n}} \\ &\quad \prod_\alpha f_{N_\alpha}\left(\frac{k_\mathrm{L}d_0\mathbf{n}_\Omega^\alpha}{2} - \frac{\mathbf{k}_\mathrm{L}^\alpha d_0}{2} - \frac{n_\alpha}{N_\alpha}\pi + m_\alpha\pi\right). \end{aligned} \quad (40)$$

Each term in the sum is labeled by the vector $\mathbf{m}$, and corresponds to a different diffraction peak. The probability that the spin-wave $\mathbf{n}$ emits a photon in the $\mathbf{m}$ diffraction peak is given by

$$p_\mathbf{n}^{[\mathbf{m}]} = \int d\Omega \ I_\mathbf{n}^{[\mathbf{m}]}(\Omega). \quad (41)$$

Eq. (40) has a clear interpretation in terms of momentum conservation. $I_\mathbf{n}(\Omega)$ has a maximum whenever there is a value of $\mathbf{m}$ such that

$$k_\mathrm{L}\mathbf{u}_\Omega^\alpha = \mathbf{k}_\mathrm{L}^\alpha + n_\alpha\frac{2\pi}{d_0 N_\alpha} + m_\alpha\frac{2\pi}{d_0}, \quad \forall \ \alpha. \quad (42)$$

That is, the linear momentum of the emitted photon has to match the sum of three contributions: the momentum of the incident laser, $\mathbf{k}_\mathrm{L}$; the initial momentum of the spin-wave, $\mathbf{n}2\pi/(d_0 N_\alpha)$, and the contribution from the lattice periodicity, which enters through the reciprocal wavevector $\mathbf{m}2\pi/d_0$. In 1D, condition (42) has to be satisfied only by the $\alpha = z$ component of these vectors, that is, only the projection of the momentum on the chain is conserved. In 3D, on the contrary, the equality has to be satisfied by all the vector components. The relation (42) determines the maxima in the emission pattern depending on d_0 and λ, but it also determines the collective rates, through the normalization condition on $I_\mathbf{n}(\Omega)$, see Eq. (30).

Since $-1 \le \mathbf{u}_\Omega^\alpha \le 1$, if $\lambda > 2d_0$ there is at most a single value of $\mathbf{m}$, for which condition (42) can be fulfilled for some vector $\mathbf{u}_\Omega$. Thus, $\lambda > 2d_0$ is the directional regime in the emission of photons, whereas in the case $\lambda < 2d_0$, we cannot ensure that photons are collimated in a single direction. In the following two subsections, we

will study these regimes in 1D and 3D. In particular, we will be interested in determining $\Delta\theta$, the angular width of the photon beam in the directional regime, and $\Gamma_\mathbf{n}$, the collective emission rates. These quantities will be studied as a function of dimensionality, d_0, λ, and N.

B. Atom chains

We consider for concreteness that $\mathbf{k}_\mathrm{L}$ points in the z direction, parallel to the chain axis. The vectors $\mathbf{n}$, $\mathbf{m}$ are reduced now to scalars n, m, that correspond to the projections on the chain axis. Due to the symmetry of the problem, the photon distribution emitted by a spin-wave n depends only on the angle θ, through $\mathbf{u}_\Omega^z = \cos\theta$,

$$\begin{aligned} I_n(\Omega) &= \\ &\frac{1}{4\pi}\frac{\bar{\Gamma}}{\Gamma_n}\frac{1}{N}\frac{\sin^2\left((k_\mathrm{L}d_0(\mathbf{u}_\Omega^z - 1) - n\frac{2\pi}{N})N/2\right)}{\sin^2\left((k_\mathrm{L}d_0(\mathbf{u}_\Omega^z - 1) - n\frac{2\pi}{N})/2\right)}. \end{aligned} \quad (43)$$

Taking the limit $N \gg 1$ and using (38),

$$\begin{aligned} I_n(\Omega) &= \sum_m I_n^{[m]}(\Omega), \quad (44) \\ I_n^{[m]}(\Omega) &= \frac{1}{4\pi}\frac{\bar{\Gamma}}{\Gamma_n}f_N\left(\frac{k_\mathrm{L}d_0}{2}(\mathbf{u}_\Omega^z - 1) - n\frac{\pi}{N} + m\pi\right). \end{aligned}$$

The peaks in the photon distribution correspond to the emission of photons such that the z component of momentum is conserved,

$$\mathbf{u}_\Omega^z - 1 = \frac{\lambda}{d_0}\frac{n}{N} - \frac{\lambda}{d_0}m, \quad (45)$$

that is, there is a maximum whenever there is a value of m which satisfies this relation. In the directional regime, $\lambda > 2d_0$, there is at most a single value of m which satisfies (45). However, in this case, momentum conservation only determines the value of θ at the emission maximum, which implies that, in general, photons are emitted in cones spanned by different values of ϕ. Only when the maximum happens at $\theta = 0$, or $\theta = \pi$, photons are collimated in the forward- or backward-scattering directions, respectively (see Fig. 2).

Let us study first the case $n = 0$, that is, the emission properties of the completely symmetric state. The forward-scattering contribution, $I_0^{[0]}$, has an angular width given by $\Delta\theta_{1D} = 1/\sqrt{k_\mathrm{L}d_0 N}$. The contribution to the emission pattern from each of the Bragg terms is

$$\begin{aligned} p_0^{[0]} &= \int d\Omega I_0^{[0]}(\Omega) = \frac{\bar{\Gamma}}{\Gamma_0}\frac{\lambda}{4d_0}, \\ p_0^{[m]} &= 2p_0^{[0]}, \quad m \ne 0. \end{aligned} \quad (46)$$

Condition (45) leads to the result that the number of emission cones with $m \ne 0$ is given by $\text{int}(2d_0/\lambda)$. A calculation made without resorting to the planewave approximation, yields the same results, even for relatively
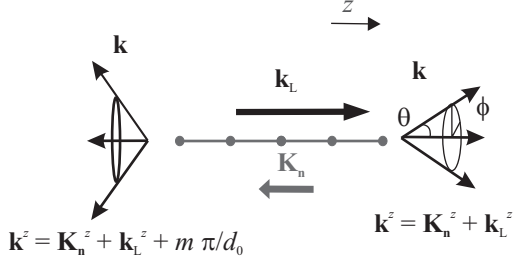
FIG. 2: Photon emission by a spin-wave with linear momentum $\mathbf{K}_n$ in an atomic chain. The maxima in the photon distribution are at angles θ such that momentum along the chain is conserved, up to reciprocal lattice wave vectors.

small atom numbers ($N = 20$), see Fig. 3. However, the shape of the photon angular distribution in the exact calculation shows a departure from the sinc^2-shape predicted by Eq. (45), see Fig. 4. By using the normalization condition for $I_0(\Omega)$, we determine the probability of emission in the forward-scattering direction,

$$p_0^{[0]} = \frac{1}{1 + 2\, \mathrm{int}\left(\frac{2d_0}{\lambda}\right)}. \tag{47}$$

Thus, in the directional regime, $\lambda > 2d_0$, all the diffraction peaks but the forward-scattering one are suppressed. The collective emission rate is given by

$$\frac{\Gamma_0}{\bar{\Gamma}} = \frac{\lambda}{4d_0} + \frac{\lambda}{2d_0}\mathrm{int}\left(\frac{2d_0}{\lambda}\right). \tag{48}$$
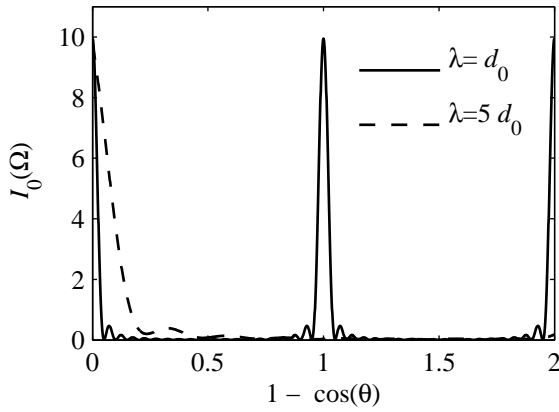

FIG. 3: Angular photon distribution of the photonic mode emitted by the completly symmetric spin-wave in a chain of $N = 20$ atoms. We have used the exact egenvalues of the density matrix, and followed the method described in the previous section.

Finally, we calculate the emission rate for all the 1D spin-wave states. They can be written as an integration over the contributions coming from different diffraction
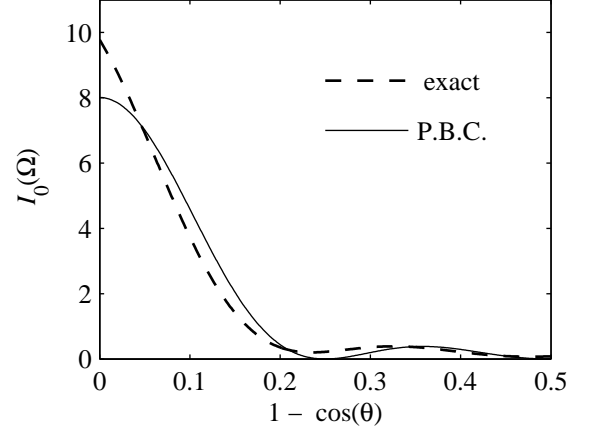

FIG. 4: Angular photon distribution of the photonic mode emitted by the completly symmetric spin-wave from a chain of $N = 20$ atoms, $\lambda > 5d_0$, calculated assuming periodic boundary conditions (continuous line), and performing and exact diagonalization of the master equation (dashed line). The distribution in the exact case shows a departure form the sinc^2-shape due to collective effects.

peaks, through the normalization condition, we have

$$\frac{\Gamma_n}{\bar{\Gamma}} = \frac{1}{4\pi}\sum_m \int_{-1}^1 dx f_N\left(\frac{k_{\mathrm{L}}d_0}{2}(x-1) - n\frac{\pi}{N} + m\pi\right). \tag{49}$$

The collective rates are thus determined by the number of diffraction peaks which appear in the emission pattern of each collective state n. Considering the limit in which $k_{\mathrm{L}}d_0 N \gg 1$, that is, the delta limit for each of these peaks, we get the following distribution of emission rates,

$$\frac{\Gamma_n}{\bar{\Gamma}} = \chi_{1D}\left((\theta(-\frac{n}{N})\theta(\frac{2d_0}{\lambda} + \frac{n}{N}) + \mathrm{int}(\frac{2d_0}{\lambda} + \frac{n}{N})\right), \tag{50}$$

with $\chi_{1D} = \lambda/(2d_0)$, and $\theta(x)$ is the Heaviside function. This expression describes quite well the emission rates calculated without assuming periodic boundary conditions, that is, by diagonalizing the matrix J_{ij} given by (14), see Fig. 5. In the limit $d_0 \gg \lambda$, we recover $\Gamma_n = \bar{\Gamma}$. On the contrary if λ is comparable to d_0, emission rates are renormalized. In the directional regime, $\lambda > 2d_0$, some of the states have an enhanced rate, $\Gamma_n = \chi_{1D}$ (superradiant), whereas there are states for which $\Gamma_n = 0$ (subradiant) [17]. Due to the relation between the photon distribution and the emission rates (49), subradiant states correspond to spin-waves whose emission pattern is not peaked at a given value of θ. On the contrary, superradiant states are spin-waves whose emission pattern does contain a maximum as a function of θ. Note that

Eq. (50) does not describe the case of Dicke superradiance, which would predict a single superradiant state, with $\Gamma_0 = N\bar{\Gamma}$. The reason is that we have assumed condition $Nk_{\mathrm{L}}d_0 \gg 1$, that is, where are always in the regime in which the light wavelength is much smaller than the size of the chain. Our results for 1D are consistent with Ref. [18], where the collective light emission form an atomic chain was studied with the quantum jump formalism.
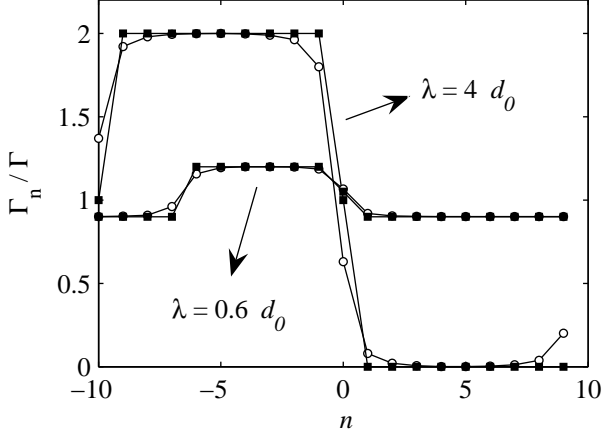

FIG. 5: Calculation of the collective rates of a chain with $N = 20$ atoms. Filled squares: result predicted by the planewave approximation (50). Empty circles: eigenvalues of the matrix J_{ij} defined by (14). In the last case, to each eigenvalue of J_{ij} we assign a wavenumber n that corresponds to the maximum Fourier component of the corresponding eigenvector.

To summarize the situation in 1D, in the directional regime $\lambda > 2d_0$, superradiant spin-waves emit photons into a single emission cone, with a rate that is enhanced by χ_{1D}. This effect can be used to generate photons that are collimated along the chain axis. Note that, in general, the atom-photon mapping induced with the Λ-scheme of Fig. 1 may compete with other radiative processes, such as the radiative decay from $|a\rangle$ back to $|s\rangle$, or the radiative decay from $|a\rangle$ to other atomic levels that are not included in the Λ-scheme. This problem can be solved by enhancing the atom-photon mapping rate, choosing $\chi_{1D} \gg 1$. Since χ_{1D} does not depend on the chain size, this implies to choose $\lambda \gg d_0$. Another way out of this problem is to use a cycling transition (see Appendix D).

C. 3D Atom lattices

Contrary to the 1D case, in 3D it is not simple to obtain closed expressions to describe the emission in the planewave approximation. In order to get a simpler picture, we replace the sinc-function in the definition (38) by a gaussian which is normalized in the same way,

$$f_N(x) \approx \sqrt{\pi}N_x e^{-x^2 N^2}. \tag{51}$$

This approximation is justified in the limit in which the diffraction peaks are narrow enough, such that they do not overlap, that is, $N_\alpha k_{\mathrm{L}}d_0 \gg 1$. Assuming that the length of the lattice is the same in any spatial direction, $N_x = N_y = N_z$, then the angular photon distribution for a state $\mathbf{n}$ is given by

$$I_{\mathbf{n}}(\Omega) = \sum_{\mathbf{m}} I_{\mathbf{n}}^{[\mathbf{m}]}(\Omega), \tag{52}$$

$$I_{\mathbf{n}}^{[\mathbf{m}]}(\Omega) = \frac{1}{4\pi}\frac{\bar{\Gamma}}{\Gamma_{\mathbf{n}}}\pi^{3/2}Ne^{-\left(\frac{k_{\mathrm{L}}d_0}{2}\mathbf{u}_\Omega - \frac{k_{\mathrm{L}}d_0}{2} - \pi\frac{\mathbf{n}}{N} - \pi\mathbf{m}\right)^2 N_x^2}.$$

A given term $I_{\mathbf{n}}^{[\mathbf{m}]}(\Omega)$, has a non-negligible contribution only if

$$\frac{|\mathbf{k}_{\mathrm{L}}d_0 + 2\pi\mathbf{n}/N + 2\pi\mathbf{m}|}{k_{\mathrm{L}}d_0} = 1, \tag{53}$$

since otherwise, there are no photons which satisfy the energy-momentum conservation.

Let us study first the directional regime, $\lambda > 2d_0$. In this case we find the following two possible situations:

(i) Superradiant states – Spin-waves for which there exists a value of $\mathbf{m}$, say $\mathbf{m}_c$, such that (53) is satisfied. The condition $\lambda > 2d_0$ ensures that there is a single value $\mathbf{m}_c$, such that the photon distribution can be simplified,

$$I_{\mathbf{n}}(\Omega) = I_{\mathbf{n}}^{[\mathbf{m}_c]}(\Omega). \tag{54}$$

Thus, there is a single emission peak in the direction,

$$\mathbf{u}_{\Omega_{\max}} = \frac{\mathbf{k}_{\mathrm{L}}d_0 + 2\pi\mathbf{n}/N + 2\pi\mathbf{m}_c}{k_{\mathrm{L}}d_0}. \tag{55}$$

Photons are collimated in a beam with width $\Delta\theta_{3D} = 1/(k_{\mathrm{L}}d_0 N^{1/3})$. From (54), and the normalization condition, we can determine the emission rate,

$$\frac{\Gamma_{\mathbf{n}}}{\bar{\Gamma}} =$$
$$\frac{N}{4\pi}\int d\Omega\, \pi^{3/2}e^{-\left(\frac{k_{\mathrm{L}}d_0}{2}\mathbf{n}_\Omega - \frac{k_{\mathrm{L}}d_0}{2} - \pi\frac{\mathbf{n}}{N} - \pi\mathbf{m}_c\right)^2 N_x^2} =$$
$$\frac{1}{\sqrt{\pi}}N_x\left(\frac{\lambda}{2d_0}\right)^2 = \chi_{3D}. \tag{56}$$

Thus, superradiant states have a decay rate that is enhanced by the optical thickness, χ_{3D}.

(ii) Subradiant states – Spin-waves for which there is no value of $\mathbf{m}$ which satisfies (53). In this case, there is not a single dominant contribution $\mathbf{m}$, such that, directionality in the emission of photons is not guaranteed. Also, the normalization condition on $I(\Omega)$ leads directly to the result $\Gamma_{\mathbf{n}} = 0$ [17].

From this analysis, we conclude that superradiant spin-waves are interesting in the context of quantum information, since they emit collimated photons with an enhanced emission rate. However the emission direction is determined by a value of $\mathbf{m}_c$ which has to be calculated for each particular spin-wave. There are two

relevant cases in which this situation becomes simpler, since energy-momentum conservation ensures that the only Bragg contribution is $\mathbf{m_c = 0}$. The first one is the case $\lambda > 4d_0$, because

$$|k_L \mathbf{u}_\Omega^\alpha - \mathbf{k}_L^\alpha| < 2k_L,$$
$$|\frac{n_\alpha}{N}\frac{2\pi}{d_0} + m_\alpha\frac{2\pi}{d_0}| > \frac{1}{2}\frac{2\pi}{d_0} \quad (\text{if } \mathbf{m} \neq \mathbf{0}), \quad (57)$$

and thus for values $\lambda > 4d_0$ the energy-momentum conservation condition (42) is only fulfilled by $\mathbf{m_c = 0}$. Second, assume that the spin-wave is created by the absorption of a photon within the Λ-configuration of Fig. 1, such that the spin-wave linear momentum can be written like $\mathbf{K_n} = \mathbf{q} - \mathbf{k}_L$, with $q = k_L$. Then, the energy-momentum condition is now

$$\mathbf{q} + \mathbf{k} = \mathbf{m}\frac{2\pi}{d_0}, \quad (58)$$

which is satisfied only by $\mathbf{m_c = 0}$ if $\lambda > 2d_0$. Indeed, the spin-wave created in this way is always superradiant, since $\mathbf{K_n}$ automatically satisfies Eq. (53). A similar result is obtained in Ref. [19] for the case of a cloud of atoms, where the rate of spontaneous emission of a state created by the absorption of a photon, is shown to increase with the optical thickness.

To summarize the situation in 3D, in the directional regime $\lambda > 2d_0$, superradiant spin-waves emit collimated photons with a rate enhanced by χ_{3D}. Contrary to the situation found in 1D, in 3D lattices $\chi_{3D} \propto N_x$, and thus it can be increased by increasing the size of the system. In this way, the atom-photon mapping could be much faster than other competing decay channels.

IV. EMISSION OF LIGHT BY HOT ATOMIC ENSEMBLES

The formalism presented in the previous section can be extended to situations in which atoms are not at fixed positions in space. In this case, one expects the photonic state not to be a pure state, but a mixed one. This is relevant to the description of hot atomic vapors. We follow the discussion presented in subsection II D, and consider the case of fast motion ($\tau_{\text{mot}} \ll \Gamma_n$). The results presented here are in agreement with Ref. [20], where the photonic mode emitted by an atomic ensemble is studied. Our first task is to reformulate the problem of photon emission by finding the master equation that describes this situation.

A. Master Equation for atomic ensembles

In the fast motion limit, particle positions may be described as a set of independent random variables, $\{\mathbf{r}_j\}$.

We choose for simplicity a gaussian distribution probability,

$$\rho(\mathbf{r}) = \frac{1}{\pi^{3/2}L^3}e^{-(r/L)^2}. \quad (59)$$

The particle positions fulfill the following identity, which will turn out to be the basis for the following calculations,

$$\langle e^{-i\mathbf{q}(\mathbf{r}_j - \mathbf{r}_l)}\rangle_{\text{mot}} = \delta_{jl} + (1 - \delta_{jl})e^{-q^2L^2/2}, \quad (60)$$

where $\langle\ldots\rangle_{\text{mot}}$ is an average over the atomic positions.

To calculate the coefficients of the master equation we start from Eq. (12) and perform the following average,

$$J_{ij} = \pi \sum_{\mathbf{k}} g_k^2 \, \delta(\omega_k - \omega_L)\langle e^{i(\mathbf{k} - \mathbf{k}_L)(\mathbf{r}_i - \mathbf{r}_j)}\rangle_{\text{mot}}, \quad (61)$$

were we have neglected the Cauchy principal value contribution, since it leads to an energy shift that does not play any role in the discussion that follows. After using (60), integrating over $\mathbf{k}$, and taking the limit $k_L L \gg 1$, we get

$$J_{ii} = \bar{\Gamma}/2,$$
$$J_{ij} = \frac{\bar{\Gamma}}{4(k_L L)^2}. \quad (i \neq j). \quad (62)$$

One can readily diagonalize the matrix J_{ij} and obtain the eigenspaces of the master equation. The first is the completely symmetric state ($n = 0$),

$$\mathcal{M}_{j0} = \frac{1}{\sqrt{N}}, \quad J_0 = \frac{\bar{\Gamma}}{2}(\chi_{\text{en}} + 1), \quad (63)$$

where $\chi_{\text{en}} = (N-1)/(2(k_L L)^2)$ is the optical thickness of the atomic ensemble. Note that in the limit $N \gg 1$, and defining the atom density, $n_{\text{at}} = N/L^3$, the optical thickness can be recast in the more familiar form $\chi_{\text{en}} = n_{\text{at}}\lambda^2 L/2$. The second eigenspace is spanned by the spin-waves orthogonal to $\mathcal{M}_{j0}$,

$$\sum_j \mathcal{M}_{jn} = 0, \quad J_n = \frac{\bar{\Gamma}}{2}\left(1 - \frac{1}{2(k_L L)^2}\right), \quad \text{if } n \neq 0. \quad (64)$$

Collective effects happen if $\chi_{\text{en}} \gg 1$. In this case, there is a single superradiant spin-wave mode, corresponding to the completely symmetric state, and $N - 1$ states which decay with the single atom emission rate, $J_n = \bar{\Gamma}/2$.

B. Single photon state

The spin-wave mode $n = 0$ is the only one to show collective effects, and it is the collective state which can be created in experiments with atomic vapors. For these reasons we focus on the following on its properties. First, we consider the emission of light by the initial atomic state with a single excitation,

$$|\Psi_0\rangle_{\text{at}} = \frac{1}{\sqrt{N}}\sum_j b_j^+|0\rangle_{\text{at}}. \quad (65)$$

The atom-photon mapping can be extended to this situation by considering first that the atomic state is mapped into a given photonic state, and then by performing the average on the atomic positions. This gives as a result a photon density matrix,

$$|\Psi_0\rangle_{\text{at}}|0\rangle_{\text{ph}}\langle 0| \;\rightarrow\; |0\rangle_{\text{at}}\rho,$$
$$\rho \;=\; \sum_{\mathbf{k},\mathbf{k}'}\rho_{\mathbf{k}\mathbf{k}'}a_{\mathbf{k}}^+|0\rangle_{\text{ph}}\langle 0|a_{\mathbf{k}'}. \qquad (66)$$

Note that the mapping is now form pure to mixed states. Unfortunately, with the statistical properties of the atomic positions considered here, it is not possible to ensure that this is still a gaussian map. We can, however, use Eqs. (24,60) to get

$$\rho_{\mathbf{k}\mathbf{k}'} \;=\; \langle\phi_{0,\mathbf{k}}\phi_{0,\mathbf{k}'}^*\rangle_{\text{mot}} = (1-\epsilon)\bar{\phi}_{0,\mathbf{k}}\bar{\phi}_{0,\mathbf{k}'}^* + \epsilon\bar{\rho},$$
$$\epsilon \;=\; \frac{1}{1+\chi_{\text{en}}}. \qquad (67)$$

We work, for clarity, in the interaction picture with respect to H_{lm}. The coherent component of the photon density matrix is the pure state $|\bar{\phi}_0\rangle_{\text{ph}}$,

$$\bar{\phi}_{0,\mathbf{k}} = \sqrt{1+\frac{1}{\chi_{\text{en}}}}\sqrt{N-1}g_k\frac{e^{-|\mathbf{k}-\mathbf{k}_{\text{L}}|^2L^2/4}}{-i(\omega_k-\omega_{\text{L}})+J_0}, \qquad (68)$$

whereas $\bar{\rho}$ is a normalized mixed state which describes the incoherent (isotropic) component,

$$\bar{\rho}_{\mathbf{k}\mathbf{k}'} = \frac{(1+\chi_{\text{en}})g_kg_{k'}e^{-|\mathbf{k}-\mathbf{k}'|^2L^2/4}}{(-i(\omega_k-\omega_{\text{L}})+J_0)(i(\omega_{k'}-\omega_{\text{L}})+J_0)}. \qquad (69)$$

Under the condition $\chi_{\text{en}} \gg 1$, the main contribution to the photon density matrix is the pure one, $|\phi_0\rangle_{\text{ph}}$. The parameters which describe the directionality in the photon emission can be readily evaluated. The emission pattern has both coherent and incoherent contributions,

$$I_0(\Omega) \;=\; I_0^{\text{coh}}(\Omega) + I_0^{\text{inc}}(\Omega),$$
$$I_0^{\text{coh}}(\Omega) \;=\; (1-\epsilon)\frac{N-1}{4\pi\chi_{\text{en}}}e^{-(k_{\text{L}}\mathbf{u}_\Omega-\mathbf{k}_{\text{L}})^2L^2/2},$$
$$I_0^{\text{inc}}(\Omega) \;=\; \epsilon\frac{1}{4\pi}. \qquad (70)$$

Thus, photons are collimated in the forward-scattering direction, with angular width $\Delta\theta_{\text{en}} = 1/(k_{\text{L}}L)$. The deviation of the photon emission from directionality is determined by the probability of emission out of the forward-scattering cone, $\mathcal{E} = \int d\Omega I_0^{\text{inc}}(\Omega) = \epsilon$.

Finally, we calculate the purity of the photon state which is defined as

$$\mathcal{P} \;=\; \text{Tr}(\rho^2)$$
$$= (1-\epsilon)^2 + \epsilon^2\text{Tr}\left(\bar{\rho}^2\right) + 2\epsilon(1-\epsilon)\langle\bar{\phi}_0|\bar{\rho}|\bar{\phi}_0\rangle. \qquad (71)$$

This quantity is very relevant when using photons in quantum information processing, since it describes the efficiency in the process of interference of photonic modes

in a beam-splitter, being $\mathcal{P} = 1$, the case corresponding to a pure state. To calculate $\mathcal{P}$, we notice first that

$$\text{Tr}\left(\bar{\rho}^2\right) = \sum_{\mathbf{k},\mathbf{k}'}|\rho_{\mathbf{k},\mathbf{k}'}|^2 = \frac{1}{2(k_{\text{L}}L)^2}. \qquad (72)$$

Also because of the Cauchy-Schwartz inequality, this implies that

$$|\langle\phi_0|\bar{\rho}|\phi_0\rangle|^2 \leq \text{Tr}\left(\bar{\rho}^2\right), \qquad (73)$$

such that,

$$\mathcal{P} = \text{Tr}(\rho^2) = (1-\epsilon)^2 + \mathcal{O}(1/k_{\text{L}}L) \approx 1 - 2/\chi_{\text{en}}. \qquad (74)$$

Thus, in the limit $k_{\text{L}}L \gg 1$, the purity of the photon state is solely determined by χ_{3D}.

C. Multiphoton state

We study now the properties of the multiphoton case. The mapping is from a state with M atomic excitations in the completely symmetric state to a multiphoton mixed state,

$$\frac{1}{\sqrt{M!}}\left(\frac{1}{\sqrt{N}}\sum_j b_j^+\right)^M|0\rangle_{\text{at}}|0\rangle_{\text{ph}} \rightarrow |0\rangle_{\text{at}}\rho. \qquad (75)$$

We assume the low-excitation limit ($M \ll N$). In Appendix B, we show that the mixed photonic state can be written like

$$\rho \;=\; (1-\epsilon)^M|\Phi\rangle_{\text{ph}}\langle\Phi| + \tilde{\rho},$$
$$|\Phi\rangle_{\text{ph}} \;=\; \frac{1}{\sqrt{M!}}\left(\sum_{\mathbf{k}}\bar{\phi}_{0,\mathbf{k}}a_{\mathbf{k}}^+\right)^M|0\rangle_{\text{ph}}. \qquad (76)$$

That is, it is a sum of a pure state consisting of M photons in the photonic mode defined by (68), and the (not normalized) mixed state $\tilde{\rho}$. In order to determine the purity of ρ^{ph} we should study the mixed contribution, which is far more complicated than in the single photon case. Insted, we notice that

$$\mathcal{P} = \text{Tr}\left(\rho^2\right) \geq (1-\epsilon)^{2M}, \qquad (77)$$

which allows us to obtain a lower bound for the purity.

V. IMPLEMENTATIONS

A. Collective light emission in ultracold atoms in optical lattices

Ultracold atoms in optical lattices are an ideal system for the observation of the effects described in this work. For example, atoms in a Mott phase [21] would

be ideally suited to study collective light emission from a square lattice. In this setup, atoms are placed at distances that are comparable to optical wavelengths, since potential wells in a standing–wave are indeed separated by $d_0 = \lambda_{sw}/2$, with λ_{sw}, the wavelength of the counterpropagating lasers that create the lattice. The conditions for the directional regime are met by using an optical transition such that $\lambda > \lambda_{sw}$. Under this conditions, ultracold atoms in optical lattices are ideal to form an atom-light quantum interface [11, 22, 23], where the quantum state of light can be indeed manipulated, as show recently in [24]. The properties of the emitted light may also be used to measure properties of quantum many-body phases in optical lattices [25]. Our theory can also be applied to the collective matter-wave emission in optical lattices proposed in Ref. [26].

We propose now an experiment in which the renormalization of the collective rates may be observed by performing the following steps (see Fig. 6):
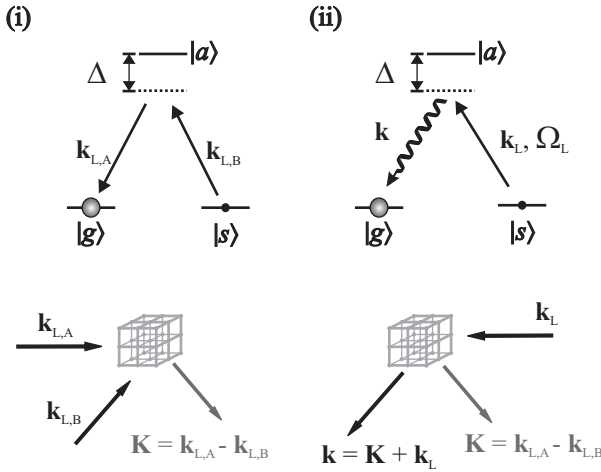

FIG. 6: (i) Creation of a spin-wave with linear momentum $\mathbf{k}_{L,A} + \mathbf{k}_{L,B}$ in an optical lattice. (ii) Mapping from the spin-wave to a photonic mode.

(i) First, we use a protocol to initialize the atomic quantum state of the lattice. Assume that all the atoms are initially in the ground state $|g\rangle$. The Λ-scheme of Fig. 1 may be used for the initialization by shining the lattice during a short time T with two lasers with wavevectors $\mathbf{k}_{L,A}$, $\mathbf{k}_{L,B}$, and Rabi frequencies $\Omega_{L,A}$, $\Omega_{L,B}$, and detuning Δ, such that $\Omega_{L,A}, \Omega_{L,B} \ll \Delta$. Under these conditions, a coherent state of spin-waves with momentum $\mathbf{K} = \mathbf{k}_{L,A} - \mathbf{k}_{L,B}$ is created,

$$
\begin{aligned}
|\Psi\rangle_{at} &\propto e^{-i(\Omega_{AB}T/2)b_K^+}|0\rangle_{at}, \\
b_{\mathbf{K}}^+ &= \frac{1}{\sqrt{N}} \sum_j e^{i\mathbf{K}\mathbf{r}_j} b_j^+,
\end{aligned} \tag{78}
$$

where $\Omega_{AB} = \Omega_{L,A}\Omega_{L,B}/\Delta$, and condition $\Omega_{AB}T \ll 1$ has to hold in order to ensure that the system is in the low excitation regime.

(ii) After the creation of the collective atomic state, this can be realeased by a second laser with momentum $\mathbf{k}_L$ in the Λ-scheme considered along this work. The analysis presented in section III can be applied to study the emission of photons by setting $\mathbf{K_n} = \mathbf{k}_{L,A} - \mathbf{k}_{L,B}$. For example, if $\mathbf{k}_L = \mathbf{k}_{L,B}$, then the spin-wave emits a photon in the direction $\mathbf{k}_{L,A}$, with a superradiant rate $\chi_{3D}\bar{\Gamma}$. On the contrary, if $\mathbf{K_n}$ does not satisfy the condition (53), then the emission rate will be suppressed, up to finite size effects.

B. Deterministic generation of entangled states of photons

An application of our ideas to generate photons in a deterministic way requires a system where experimentalists are both able to reach the directional regime of collective light emission, and to initialize the atomic system in a given spin-wave state. This idea may find useful applications in quantum cryptography [27], quantum computation [28], and quantum litography [29].

Before going into the description of particular experimental setups, we discuss how to extend our formalism to generate photons that are entangled in polarization. For this, we consider the double-Λ scheme of Fig. 7, in which there are two excited levels, $|s_1\rangle$, $|s_2\rangle$, and a ground state level $|g\rangle$. The atom-photon mapping is described in the same way, by defining the corresponding Holstein-Primakoff operators $b_{1,j}$, $b_{2,j}$. Assume that the $|s_1\rangle$, $|s_2\rangle$ states decay by emitting photons with different polarizations and creation operators $a_{1,\mathbf{k}}^+$, $a_{2,\mathbf{k}}^+$. If the conditions for momentum conservation explained along this work are fulfilled, then entangled spin waves will be mapped into entangled photons in polarization. For example, consider the following initial spin-wave state,

$$
|\Psi\rangle_{at} = \frac{1}{\sqrt{2}} \left(b_{1,\mathbf{K}_A}^+ b_{2,\mathbf{K}_B}^+ + b_{1,\mathbf{K}_B}^+ b_{2,\mathbf{K}_A}^+ \right) |0\rangle_{at}, \tag{79}
$$

where we have used the notation $b_{\sigma,\mathbf{K}}^+ = (1/\sqrt{N})\sum_j e^{i\mathbf{K}\mathbf{r}_j} b_{\sigma,j}^+$. After the atom-photon mapping, the following photonic state is created,

$$
\begin{aligned}
|\Phi\rangle_{ph} = \frac{1}{\sqrt{2}} \\
\left(a_{1,\mathbf{K}_A+\mathbf{k}_L}^+ a_{2,\mathbf{K}_A+\mathbf{k}_L}^+ + a_{1,\mathbf{K}_B+\mathbf{k}_L}^+ a_{2,\mathbf{K}_A+\mathbf{k}_L}^+ \right) |0\rangle_{ph}.
\end{aligned} \tag{80}
$$

In this way, the ability to generate entangled spin-waves is equivalent, by virtue of the atom-photon mapping, to the ability to generate entangled photonic states. We propose three experimental set-ups, where this is possible and discuss the conditions that are required for the implementation of this idea.
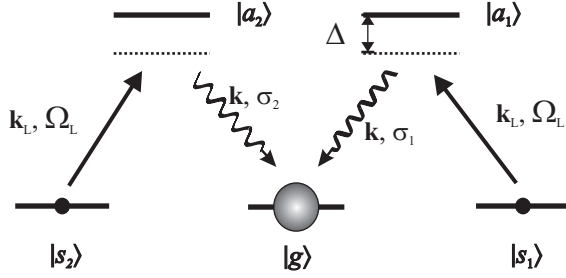
FIG. 7: Double Λ-scheme for the generation of photons entangled in polarization.

FIG. 8: Angular photon distribution of the photonic mode emitted by the completely symmetric spin-wave in an ion chain with $N = 10$ ions.

1. Trapped ions

This system is specially appealing from the point of view of the creation of entangled states, since quantum gates allow us to create deterministically any collective state. This idea has been demonstrated in the creation of the $\mathbf{K} = \mathbf{0}$ (W) state, in [4]. The first issue that we have to deal with, is whether conditions for forward-scattering of photons hold here. Usually ions are arranged in chains, with the peculiarity that the distance between ions is not constant. However, if condition $d_0^{\mathrm{av}} < \lambda/2$, with d_0^{av} the average distance, we still get light emission in the forward–scattering cone only. We show this fact by performing a calculation of the emission pattern of a single atomic excitation in the $\mathbf{K} = \mathbf{0}$ state, using the theory presented in the previous sections, see Fig. 8. Note that directionality is achieved even for a relatively small number of ions ($N = 10$). The main difficulty for the implementation of this idea with ions lies on the fact that ion–ion distances are usually in the range of a few μm, and thus condition $d_0^{\mathrm{av}} < \lambda/2$ is not fulfilled when considering optical wavelengths [30]. A way out of this problem is to use optical transitions which lie in the range of $\lambda \gtrsim 5\mu m$, like for example $\lambda(^2\mathrm{D}_{3/2} \text{ - } ^2\mathrm{P}_{1/2}) = 10.8$ μm in Hg$^+$, or $\lambda(^2\mathrm{D}_{3/2} \text{ - } ^2\mathrm{D}_{5/2}) = 12.5$ μm in Ba$^+$. A particularly interesting initial atomic state is the spin-wave (79) with $\mathbf{k}_A = \mathbf{0}$, and $\mathbf{k}_B = -2\mathbf{k}_{\mathrm{L}}$, (all vectors in the direction of the atomic chain), since it leads to the emission of an entangled photon pair in the forward and backward-emission directions.

2. Ultracold atoms in optical lattices

We have shown above that optical lattices are well suited to reach the regime of directionallity in the emission of photons. The main issue here, contrary to the case of trapped ions, is to find a way to create efficiently the initial spin-wave states in a deterministic way. Although one could think of peforming quantum gates between ultracold neutral atoms to generate collective atomic states [31, 32], this procedure faces the difficulties of quantum computation in this system, like for example, how to achieve single atom addressability.
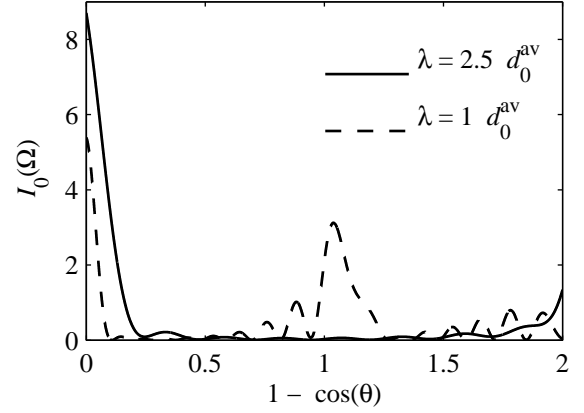
More efficiently, one could avoid the use of quantum gates by using the dipole-blockade mechanism with Rydberg atoms, which allows us to generate W-states, as well as states which emit Fock states with a number M of photons [2]. Interactions between excited atomic states, like those that take place in Rydberg atoms, can be also used to generate photons entangled in polarization. This can be achieved in a single experimental step, without the need for quantum gates, if the proper configuration of atomic interactions is chosen. As an example, consider the level configuration shown in Fig. 9, and interactions between excited states such that atoms in levels $|s_1\rangle$, $|s_2\rangle$, interact strongly only if they are in the same excited state, that is, $U_{11} = U_{22} = U$, but $U_{12} = 0$. We apply two lasers with wavevectors $\mathbf{k}_{\mathrm{A,B}}$ and Rabi frequencies $\Omega_{\mathrm{A,B}}$, detuned with respect to the $|g\rangle - |s_{a,b}\rangle$ transition, such that $\Delta_{\mathrm{A}} = -\Delta_{\mathrm{B}} = \Delta$. If condition $\Delta_{\mathrm{A,B}} \gg \Omega_{\mathrm{A,B}}$ is fulfilled, then the lasers induce a two–photon transition with Rabi frequency $\Omega_{\mathrm{eff}} = \Omega_{\mathrm{A}}\Omega_{\mathrm{B}}/\Delta$. Furthermore, if $\Omega_{\mathrm{eff}} \ll U$, states with two atoms in the same excited state are not populated. Under these conditions there are two possible excitation channels, depicted in Fig. 9, which give rise to the linear combination (79).

3. Atomic ensembles at room temperature

The very same techniques which can be applied to Rydberg atoms in an optical lattice can also be used in the case of hot ensembles. On the one hand, this setup has the advantage that atoms do not need to be cooled and placed in an optical lattice. On the other hand, it can be described by a statistical distribution of particles, and thus suffers from the fact that high efficiency in the release of photons is achieved under more severe conditions of particle density and atom number, as discussed above. However, densities which are high enough to fulfill the requirement $\mathcal{E} \ll 1$ have been recently reported in [33].
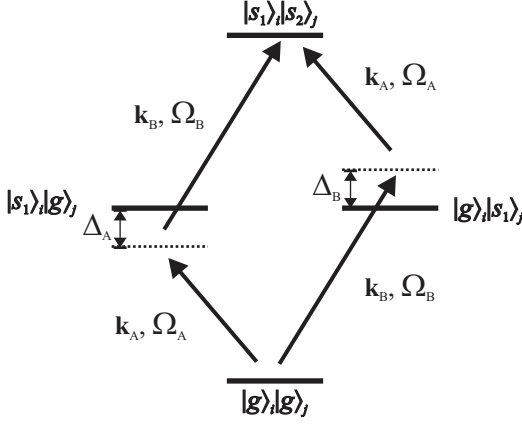
FIG. 9: Scheme for the generation of entangled spin-waves by using the Rydberg blockade.

C. Photon up-conversion

Finally, the ideas presented in this work find an intereseting application in the efficient up-conversion of photons. For example, transitions between the states $|g\rangle$ and $|s\rangle$, are typically in the range of GHz. Application of the Λ scheme of Fig. 1, provides a way for up-conversion of the microwave photon to an optical photon. Also, by using several lasers, one could induce N-photon transitions from $|g\rangle$ to $|a\rangle$, and emit a single high frequency photon. By using the ideas presented along this work, this could be done in such a way that the up-converted photons are collimated and, thus, they can be efficiently collected.

VI.   ACKNOWLEDGMENTS

APPENDIX A: VALIDITY OF NEGLECTING PROPAGATION EFFECTS

When performing the integration in k in Eq. (27), we have to deal with the following factor inside the integration,

$$\frac{e^{i(\mathbf{k}-\mathbf{k}_{\mathrm{L}})(\mathbf{r}_j-\mathbf{r}_{j'})}}{\left(-i(\omega_k-\omega_{\mathrm{L}})-\bar{J}_n^*\right)\left(i(\omega_k-\omega_{\mathrm{L}})-\bar{J}_{n'}\right)}, \qquad (A1)$$

which can be evaluated in the following way,

$$\frac{1}{\left(-i(\omega_k-\omega_{\mathrm{L}})-\bar{J}_n^*\right)\left(i(\omega_k-\omega_{\mathrm{L}})-\bar{J}_{n'}\right)} =$$
$$-\left(\frac{1}{-i(\omega_k-\omega_{\mathrm{L}})-\bar{J}_n^*}+\frac{1}{i(\omega_k-\omega_{\mathrm{L}})-\bar{J}_{n'}}\right)/(\bar{J}_n^*+\bar{J}_{n'})$$
$$\approx -2\pi\delta(\omega_k-\omega_L)\frac{1}{\bar{J}_n^*+\bar{J}_{n'}},$$

where he have aproximated

$$\mathrm{Re}\left(\frac{1}{-i(\omega_k-\omega_{\mathrm{L}}-\Delta_n)-\Gamma_n/2}\right)\approx\pi\delta(\omega_k-\omega_{\mathrm{L}}). \,(A2)$$

This approximation is justified by the following argument. The left-hand side of (A2) is a function of k with width Γ_n/c. The latter has to be compared with the width of the exponential in (A1), which is, roughly $1/L$. Thus, the approximation holds in the limit $\Gamma_n/c \ll 1/L$, that is, whenever the emission time is shorter than the propagation of the photon through the sample, something that is well justified in the trapping setups considered in this work. The energy shift can also be safely neglected, since condition $\Delta_n \ll \omega_L$ is well justified by the atomic transitions used in cold atom setups.

APPENDIX B: MULTIPHOTON STATE EMITTED BY AN ATOMIC ENSEMBLE

Our starting point is the atom-photon mapping defined in Eq. (75). By applying our method we get

$$\rho = \frac{1}{M!}\frac{1}{N^M}\langle\left(\sum_{j,\mathbf{k}}\frac{e^{i(\mathbf{k}-\mathbf{k}_{\mathrm{L}})\mathbf{r}_j}}{i(\omega_k-\omega_{\mathrm{L}})-\Gamma_0/2}a_{\mathbf{k}}^+\right)^M$$
$$|0\rangle_{\mathrm{ph}}\langle 0|\left(\sum_{j',\mathbf{k}'}\frac{e^{-i(\mathbf{k}'-\mathbf{k}_{\mathrm{L}})\mathbf{r}_{j'}}}{-i(\omega_{k'}-\omega_{\mathrm{L}})-\Gamma_0/2}a_{\mathbf{k}'}\right)^M\rangle_{\mathrm{mot}}. (B1)$$

After expanding the two parenthesis to the power of M we can reexpress this equation as a sum of $2M$ products of exponentials of the argument $i(\mathbf{k}-\mathbf{k}_{\mathrm{L}})\mathbf{r}_j$. There are $\binom{N}{2M}$ terms such that all the random variables are different. These terms can be summed up to a contribution which yields a pure state, such that ρ is the sum of a pure and a mixed state,

$$\rho = \tilde{\rho}_{\mathrm{pure}} + \tilde{\rho}, \qquad (B2)$$

where $\tilde{\rho}_{\mathrm{pure}}$, $\tilde{\rho}$, are not normalized. The pure contribution has the form,

$$\tilde{\rho}_{\mathrm{pure}} = \frac{1}{M!}\frac{1}{N^M}\binom{N}{2M}\left(\sum_{\mathbf{k}}\frac{g_k e^{-|\mathbf{k}-\mathbf{k}_{\mathrm{L}}|^2 L^2/4}}{i(\omega_k-\omega_{\mathrm{L}})-\Gamma_0/2}a_{\mathbf{k}}^+\right)^M$$
$$|0\rangle_{\mathrm{ph}}\langle 0|\left(\sum_{\mathbf{k}'}\frac{g_{k'}e^{-|\mathbf{k}'-\mathbf{k}_{\mathrm{L}}|^2 L^2/4}}{-i(\omega_{k'}-\omega_{\mathrm{L}})-\Gamma_0/2}a_{\mathbf{k}'}\right)^M, \qquad (B3)$$

which is a pure state with M photons. To normalize this state, we use the limit $\binom{N}{2M}/N^M = N^M, (N \gg M)$, and obtain the final result (76).

APPENDIX C: 3D LATTICE, LIMIT $\lambda \ll d_0$

In the case of atoms in a 3D square lattice, in the limit $\lambda \ll d_0$, there are in principle many diffraction peaks which contribute to the emission pattern. In the following we show, that, at least for the completely symmetric state, if the size of the system is large enough, the forward-scattering contribution is the most important one.

Recall the definition of the probability that the spin-wave $\mathbf{0}$ emits a photon in the $\mathbf{m}$ Bragg scattering peak, $p_{\mathbf{0}}^{[\mathbf{m}]}$ (41). The forward-scattering contribution $\mathbf{m} = \mathbf{0}$, is given by

$$p_{\mathbf{0}}^{[\mathbf{0}]} = \frac{\bar{\Gamma}}{\Gamma_{\mathbf{0}}}\chi_{3D}. \tag{C1}$$

We consider for concreteness $\mathbf{k}_{\mathrm{L}} = k_{\mathrm{L}}\hat{z}$. To determine the probability of scattering out of the forward-direction, $\mathcal{E}$ we calculate the sum over the contributions with $\mathbf{m} \neq \mathbf{0}$,

$$\mathcal{E} = \sum_{\mathbf{m} \neq \mathbf{0}} p_{\mathbf{0}}^{[\mathbf{m}]} =$$
$$\frac{\bar{\Gamma}}{\Gamma_{\mathbf{0}}}\sum_{\mathbf{m} \neq \mathbf{0}}\frac{N}{4\pi}\int d\Omega\, \pi^{3/2}e^{-\left(\mathbf{u}_\Omega - \hat{z} - \frac{\lambda}{d_0}\mathbf{m}\right)^2 (k_{\mathrm{L}}d_0 N_x/2)^2}. \tag{C2}$$

Considering the limit $k_{\mathrm{L}}d_0 N_x/2 \gg 1$, the angular integral yields,

$$\mathcal{E} = \sum_{\mathbf{m} \neq \mathbf{0}}\frac{\bar{\Gamma}}{\Gamma_{\mathbf{0}}}\pi^{3/2}\frac{N}{(k_{\mathrm{L}}d_0 N_x)^2}e^{-(1 - |\frac{\lambda}{d_0}\mathbf{m} + \hat{z}|)^2 (k_{\mathrm{L}}d_0 N_x/2)^2}. \tag{C3}$$

In the limit $d_0 \gg \lambda$, we can replace the sum by an integration over $\mathbf{m}$, and get the result,

$$\mathcal{E} = \frac{\bar{\Gamma}}{\Gamma_{\mathbf{0}}}. \tag{C4}$$

Together with the normalization condition $\sum_{\mathbf{m}} p_{\mathbf{m}} = 1$, this leads to

$$\mathcal{E} = \frac{1}{1 + \chi_{3D}}, \quad \frac{\Gamma_{\mathbf{0}}}{\bar{\Gamma}} = 1 + \chi_{3D}. \tag{C5}$$

Thus, as long as the optical thickness is large, the completely symmetric state is superradiant, and photons are collimated in the forward-scattering direction. Note that the regime $\lambda \ll d_0$ is very different than the directional

regime, in the sense that there is always a non-zero probability of emission out of the forward-scattering cone. In the fully directional regime ($\lambda > 2d_0$), on the contrary $\mathcal{E} = 0$, up to finite size effects.

APPENDIX D: CYCLING TRANSITION

Up to now we have neglected the decay form the auxiliary level $|a\rangle$ back to the excited state $|s\rangle$. This process can be suppressed by considering, for example, a level configuration like the one presented in Fig. 10. $|s\rangle$ and
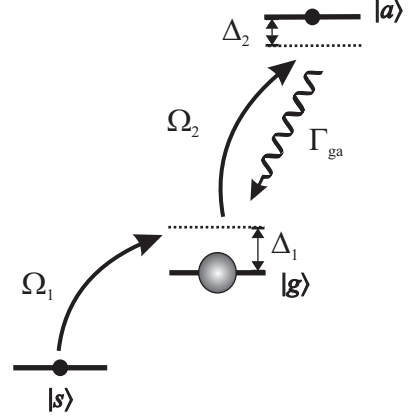

FIG. 10: Level scheme for the implementation of the atom-photon mapping, avoiding incoherent processes.

$|g\rangle$ could be two hyperfine states of the electronic ground state manifold, and $|a\rangle$ be chosen such that the $|a\rangle$-$|g\rangle$ is a cycling transition. For example, these levels could be found in ^{87}Rb: $|s\rangle \to |S_{1/2}, F = 1, m_F = 1\rangle$, $|g\rangle \to |S_{1/2}, F = 2, m_F = 2\rangle$, and $|a\rangle \to |P_{1/2}, F = 3, m_F = 3\rangle$. Spontaneous emission from $|a\rangle$ to $|s\rangle$ is forbidden by selection rules on m_F, but atoms decay from $|a\rangle$ to $|g\rangle$ with a rate Γ_{ga}. Fields with Rabi frequencies Ω_1, Ω_2, and detunings Δ_1, Δ_2, are coupled to the transitions $|s\rangle$-$|g\rangle$, $|g\rangle$-$|s\rangle$, respectively. Under some conditions, lasers induce a two-photon transition that is equivalent to a Λ-configuration such as the one considered in this work with $\Omega_{\mathrm{L}} = \Omega_{\mathrm{eff}} = \Omega_1\Omega_2/\Delta_1$.

We consider two situations:

(i) $\Delta_2 \gg \Gamma_{\mathrm{ga}}$. The atom-photon mapping is possible under conditions: $\Omega_{\mathrm{eff}} \ll \Delta_2$ (adiabatic elimination of the level $|a\rangle$), $\sqrt{N}\Omega_1 \ll \Delta_1$ (to avoid real transitions from $|s\rangle$ to $|g\rangle$), and $\Gamma_{\mathrm{ga}}(\Omega_{\mathrm{eff}}/\Delta_2)^2 \gg N\Gamma_{\mathrm{ga}}(\Omega_2/\Delta)^2$, or equivalently, $\Omega_1/\Delta_2 \gg \sqrt{N}$, to avoid the dephasing induced by the transfer of atoms from $|g\rangle$ to $|a\rangle$.

(ii) $\Delta_2 = 0$ (resonant case). Conditions: $\Omega_{\mathrm{eff}} \ll \Gamma_{\mathrm{ga}}$ (adiabatic elimination of $|a\rangle$). $\Omega_1/\Delta_2 \gg \sqrt{N}$, and $\Gamma_{\mathrm{ga}}(\Omega_{\mathrm{eff}}/\Gamma_{\mathrm{ga}})^2 \gg N\Gamma_{\mathrm{ga}}(\Omega_2/\Delta_1)$, or equivalently, $\Omega_1^2 \gg N\Gamma_{\mathrm{ga}}^2$ (same reasons as (i)).

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